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Paracompactness of paratopological groups which are GO-spaces

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ABSTRACT

Let G be a paratopological group which is a GO-space. We have showed that if the multiplication in G preserves the order on G, then G is paracompact.

1. Introduction and terminology

Recall that a *paratopological group* is a group with a topology such that the multiplication in the group is continuous. In [1], Chen has shown that a Hausdorff first countable paratopological group has a G_δ-diagonal, and in [3], Lutzer has proved that a linearly ordered space with a G_δ-diagonal is metrizable. Those two results yield to the fact that a Hausdorff first countable paratopological group which is also a linearly ordered space, is paracompact.

In this paper, we show that any paratopological group which is a GO-space such that the multiplication in the group preserves the order, has a $\mathcal{W}$ satisfying condition (F) as in the hypotheses of Corollary 2.4 in [4], and therefore it is paracompact.

Recall that a T_1 topological space X has a $\mathcal{W}$ satisfying (F), if $\mathcal{W} = \{\mathcal{W}(x)\colon x \in X\}$ where each $\mathcal{W}(x)$ consists of subsets of X containing x and

(F) if $x \in U$ and U is open, then there exists an open set V such that, $V = V(x, U)$ containing x such that $x \in W \subseteq U$ for some $W \in \mathcal{W}(y)$ whenever $y \in V$.

In [4, Corollary 2.4], it was obtained that "if the space X has a $\mathcal{W}$ satisfying (F), and if for each x there exists a non-negative integer $n(x)$ such that $\mathcal{W}(x) = \bigcup\{\mathcal{W}_i(x)\colon 0 \leqslant i \leqslant n(x)\}$ where $\mathcal{W}_i(x)$ is a chain of neighbourhoods of x with respect to inclusion for each $i \leqslant n(x)$, then X is metacompact".

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2. Main result

Let $(X, \leqslant)$ be a linearly ordered set. A subset C of X is called *convex* if $\{x \in X\colon a \leqslant x \leqslant b\}$ is a subset of C whenever a, b are points of C with $a \leqslant b$. Let $(X, \mathcal{T})$ be a topological space and "$\leqslant$" be a linear order on X. Recall that $(X, \mathcal{T}, \leqslant)$ is called *GO-space* if $\mathcal{T}$ contains the usual open interval topology on X and has a base consisting of convex subsets.

Let $(G, \mathcal{T}, \leqslant)$ be a GO-space and G be a paratopological group with the topology $\mathcal{T}$. Then we say that G is a *GO-paratopological group*. In addition, if the multiplication in G preserves order (that is, whenever $x \leqslant y$, then we have $xz \leqslant yz$ and $zx \leqslant zy$ for each element z of G), then we will say that G is an *order preserving GO-paratopological group*.

The proof of the following lemma is routine.

Lemma 2.1. *Let G be an order preserving GO-paratopological group. One of the following collections is a base of G at the neutral element e.*

(i) $\{\{e\}\}$;
(ii) $\{[e, p)\colon p > e\}$;
(iii) $\{(p, e]\colon p < e\}$;
(iv) $\{(p^{-1}, p)\colon p > e\}$.

Theorem 2.2. *Every order preserving GO-paratopological group G has a $\mathcal{W} = \{\mathcal{W}(x)\colon x \in X\}$ satisfying (F) where $\mathcal{W}(x)$ is an union of at most two chains with respect to inclusion which consists of neighbourhoods of x, for each x.*

Proof. We have four cases from the above lemma and we claim:

(i) If the collection $\{\{e\}\}$ is a base at the neutral element e in G, then for each $x \in G$,

$$\mathcal{W}(x) = \{\{x\}\}.$$

(ii) If the collection $\{[e, p)\colon p > e\}$ is a base of G at e, then for each $x \in G$,

$$\mathcal{W}(x) = \big\{\big[r^{-1}x, rx\big]\colon r > e\big\} \cup \big\{[x, xr)\colon r > e\big\}.$$

(iii) If the collection $\{(p, e]\colon p < e\}$ is a base of G at e, then for each $x \in G$,

$$\mathcal{W}(x) = \big\{\big[r^{-1}x, rx\big]\colon r > e\big\} \cup \big\{(xr, x]\colon r < e\big\}.$$

(iv) If the collection $\{(p^{-1}, p)\colon p > e\}$ is a base of G at e, then for each $x \in G$,

$$\mathcal{W}(x) = \big\{\big[r^{-1}x, rx\big]\colon r > e\big\}$$

is the collection that is required.

Case (i) holds only if G is a discrete topological space, in which case the theorem holds. For the other cases, let $x \in G$, let O be an open subset of G and let $x \in O$. Since the multiplication in the group G is continuous and it preserves the order, the set $x^{-1}O$ is an open neighbourhood of the neutral element e.

If we have (ii), then there is a $p \in G$ with $p > e$ satisfying $[e, p) \subseteq x^{-1}O$ and $[e, p)$ is an open subset of G. By continuity of the multiplication, there exists a $q \in G$ with $q > e$ and $qq \leqslant p$ such that $[e, q)[e, q) \subseteq [e, p)$. Let $V(x, O) = [x, xq)$. Take any $y \in V(x, O)$. If $y = x$, then we have $W = [y, yq) \in \mathcal{W}(y)$, otherwise we have $W = [r^{-1}y, ry] \in \mathcal{W}(y)$ where $r = yx^{-1}$. It is clear that $x \in W \subseteq O$.

Case (iii) is similar to (ii).

If we have (iv), then there is a $p \in G$ with $p > e$ satisfying $(p^{-1}, p) \subseteq x^{-1}O$. By the continuity of the multiplication, there exists a $q \in G$ with $q > e$ and $qq \leqslant p$ such that $(q^{-1}, q)(q^{-1}, q) \subseteq (p^{-1}, p)$. Let $V(x, O) = (xq^{-1}, xq)$. Take any $y \in V(x, O)$. Then we have $[r^{-1}y, ry] \in \mathcal{W}(y)$ and $x \in [r^{-1}y, ry] \subseteq O$ where

$$r = \begin{cases} xy^{-1}, & y < x, \\ yqy^{-1}, & y = x, \\ yx^{-1}, & y > x. \end{cases} \qquad \square$$

Collectionwise normality of GO-spaces, Corollary 2.4 in [4] and Theorem 2.2 give us the following.

Corollary 2.3. *Every order preserving GO-paratopological group G is paracompact.*

We recall that a subset L of a limit ordinal κ is called *stationary* in κ, if each subset of κ which is closed and unbounded in κ meets L. In [2], it was established that "a GO-space X is not paracompact if and only if there is a closed subspace of X which is homeomorphic to a stationary set in a regular uncountable cardinal". So, by [2] and Corollary 2.3, we can say that:

Corollary 2.4. *An order preserving GO-paratopological group G does not have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

References

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